Jan 2001

Day 1, Q1

Is the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ diagonalizable?

Solution.

Day 1, Q2

Evaluate $I = \oint_\Gamma f(z)dz$, where Γ is the square in the complex plane with corners $-1-i$, $1-i$, $1+i$ and $-1+i$ oriented counterclockwise, and the function $f(z)$ is

(a) $f(z) = \mathrm{Re}(z)$

Solution.

(b) $f(z) = \cosh\left(\frac{1}{2z-1}\right)$

Solution.

Day 1, Q3

Consider sequences of the form $x = (x_1, x_2, \ldots, x_n, \ldots)$, such that $\sum_{i=1}^{\infty} |x_i|^3$. Show that

$$\sum_{i=1}^{\infty} \frac{|x_i|}{n} < \infty.$$

Solution.

Day 1, Q4

Consider the Fourier integral transform pair that is defined as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+ikx} F(k)dk \quad \text{and} \quad F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$

Consider the function

$$f(x) = \begin{cases} x+1 & \text{for } -1 < x < 0 \\ 1-x & \text{for } 0 < x < 1 \end{cases}$$

(a) Sketch $f(x)$.

Solution.

(b) Compute the second derivative of $f(x)$ in the sense of distributions.

Solution.

(c) Find the Fourier transform $F(k)$ of $f(x)$.

Solution.

Day 1, Q5 Let $0 < r < 1$ and $0 < s < 1$. Consider the matrix

$$P = \begin{pmatrix} 1-r & r \\ s & 1-s \end{pmatrix}$$

Find the limit

$$Q = \lim_{n\to\infty} P^n$$

Hint: Find the eigenvalues of P and use this to show to show that the limit exists. Let $\mathbf{1}$ be the column vector with both entries equal to 1. Show that $Q\mathbf{1} = \mathbf{1}$ and that $QP = Q$. These last two equations should be enough to determine Q.

Solution.

Day 1, Q6

Find the integral

$$\int_C \frac{xdy - ydx}{x^2 + y^2}$$

along the oriented broken line C with vertices $(2, -2), (4, 4), (-5, 5)$, oriented counterclockwise.

Solution.

Day 1, Q7

Show that for the dynamical system

$$\begin{cases} \dot{x} = x(x^2 + 2y^2 - 1) \\ \dot{y} = y(y^2 + 2z^2 - 1) \\ \dot{z} = z(z^2 + 2x^2 - 1) \end{cases}$$

the sphere $x^2+y^2+y^2 = 1$ is invariant. What is the maximal domain of stability of the origin? What is the maximal of domain of stability of the origin? What happens to trajectories with $x^2 + y^2 + z^2 > 1$ at $t = 0$?

Solution.

Day 1, Q8

Consider the set of measurable functions $[0, \infty)$ with two different norms:

- $||f|| = \int_0^\infty |f(x)|dx$

- $||f||' = \int_0^\infty |f(x)|e^{-x^2}dx$

Let f_n be a sequence of functions that converges to zero uniformly on $[0,\infty)$.

(a) Show by counterexample that $||f_n||$ need not converge to zero.

Solution.

(b) Prove that $||f_n||'$ does converge to zero.

Solution.

Day 1, Q9

Estimate the number of steps needed to approximate the solution of

$$\dot{y} = 10(e^t - y) + e^t, \quad y(0) = 1,$$

on the interval $[0,1]$, with an absolute error less than 10^{-5}, using the forward Euler method

$$\frac{y_{n+1} - y_n}{h} = 10(e^{t_n} - y_n) + e^{t_n}$$

with $t_n = nh$.

Solution.

Day 1, Q10

Solve the diffusion equation

$$u_t = u_{xx} - u,$$

for $u(x,t)$ over the domain $0 \le x \le 1$ and $t \ge 0$, with the boundary conditions

$$u(0,t) = u(1,t) = 0,$$

and the initial condition

$$u(x,0) = \sin(\pi x).$$

Solution.

Day 1, Q11

The problem is to compute the function $y = x^n$ for $x > 0$ in a context where there is roundoff error. A numerical analyst proposes a two stage method: set $u = \ln(x)$ and then $y = \exp(nu)$.

(a) Find the relative error $\Delta y/y$ in $y = x^n$ in terms of a relative error $\Delta x/x$ in x. The Δx is considered small.

Solution.

(b) Find the relative error $\Delta y/y$ in $y = \exp(nu)$ in terms of a relative error $\Delta u/u$ in u. The Δu is considered small.

Solution.

(c) Again take Δx and Δu small. Suppose that the relative errors $\Delta x/x$ and $\Delta u/u$ are equal (to a good approximation). Are there values of n and x for which the relative error $\Delta y/y$ due to relative error $\Delta u/u$ in the intermediate stage is earlier than the relative error $\Delta y/y$ inherent in the problem due to relative error $\Delta x/x$ in the input. If so, find these values.

Solution.

Day 1, Q12

Find the Green's function for the operator

$$Lu = -\frac{d^2u}{dx^2} - 4u, \quad 0 \le x \le 1$$

subject to the conditions $u(0) = u'(0) = 0$.

Solution.

Day 2, Q1

Show, using the method of contour integration, that

$$\int_{-\infty}^{\infty} \frac{\cos(x) - \cos(a)}{x - a} dx = -\pi \sin(a).$$

Hint: Make use of the principal value integral.

Solution.

Day 2, Q2 For $0 < \alpha < 1$, one defines the space

$$C^\alpha([0,1]) = \left\{ u(x) \in C([0,1]) : ||u||_\alpha = \sup_{x\in[0,1]} |u(x)| + \sup_{x\in[0,1]} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \right\}.$$

Show that if $u(x), v(x) \in C^\alpha([0,1])$, then

$$u(x)\dot{v}(x) \in C^\alpha([0,1]),$$

and

$$||uv||_\alpha \le 3||u||_\alpha \dot{|}|v||_\alpha.$$

Solution.

Day 3, Q3

In this problem the Fourier transform is

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-ixy} f(x) dx$$

and the inverse Fourier transform is

$$f(x) = \int_{-\infty}^{\infty} e^{ixy} f(y) \frac{dy}{2\pi}.$$

These provide an isomorphism between the Hilbert spaces $L^2(\mathbb{R}, dx)$ and $L^2(\mathbb{R}, \frac{dy}{2\pi})$. The norm of f in the first space is equal to the norm of $\hat{f}$ in the second space.

Make the assumption that $|y| > a$ implies $\hat{f}(y) = 0$. That is, the Fourier transform of f vanishes outside of the interval $[-a, a]$. Let

$$g(x) = \frac{\sin(ax)}{ax}$$

The problem is to prove that

$$f(x) = \sum_{m=-\infty}^{\infty} f\left(\frac{m\pi}{a}\right) g\left(x - \frac{m\pi}{a}\right).$$

This says that if you know f at multiples of π/a, then you know f at all points.

Hint: Let $g_m(x) = g(x - m\pi/a)$. The task is to prove that $f(x) = \sum_m c_m g_m(x)$ with $c_m = f(m\pi/a)$. It helps to use the Fourier transform of these functions. First prove that the Fourier transform of $g(x)$ is given by $\hat{g}(y) = \pi/a$ for $|y| \le a$ and $\hat{g}(y) = 0$ for $|y| > a$. (Actually, it may be easier to deal with the inverse Fourier transform.) Then prove that $\hat{g}_m(y) = \exp(-im\pi y/a)\hat{g}(y)$. Finally, note that the functions $\hat{g}_m(y)$ are orthogonal.

Solution.

Day 2, Q4

Find the first term in a uniformly valid asymptotic expansion, as $\epsilon \to 0^+$, of the solution $u(t; \epsilon)$ to

$$\frac{d^2u}{dt^2} + u = \epsilon(u^2 - 1)\frac{du}{dt}, \quad u(0; \epsilon) = 2a > 0, \frac{du(0; \epsilon)}{dt} = 0, \ \epsilon > 0.$$

Hint: Let $u(t; \epsilon) = v(t, \tau)$ with $\tau = ct$; expand v in powers of ϵ and find the first term, $v_0(t, \tau)$, of this expansion

Solution.

Day 2, Q5

Consider the following linear multistep method for the differential equation $y' = f(t, y)$

$$y_{n+1} = 3y_n - 2y_{n-1} + \frac{h}{2}[f_n - 3f_{n-1}], \quad n = 1, 2, \ldots$$

Assume that the error in the starting value $y_0 = y(0)$ tends to zero as $h \to 0$. Find the truncation error for this method. Is this algorithm stable?

Solution.

Day 2, Q6

Find the extremal solution(s) of the functional

$$F[y] = \int_0^\pi bigl[(y')^2 + y^2 y' + x^2\big] dx,$$

subject to the constraint

$$G[y] = \int_0^\pi y^2 dx = 1$$

and boundary condition

$$y(0) = y(\pi) = 0.$$

Solution.

Day 2, Q7

The Bessel function of order n, $J_n(x)$, has the integral representation

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(t) - nt) dt.$$

Show that $J_n(n) \propto n^\alpha$ as $n \to \infty$, and find the exponent α.

Hint: You may wish to use the result

$$\int_0^\infty e^{-s^p} ds = \frac{1}{p} \Gamma\big(\frac{1}{p}\big).$$

Solution.

Day 2, Q8

Show that there is a continuous real function F defined on the unit sphere $S = \{u : ||u||^2 = 1\}$ in real Hilbert space that does not assume its maximum.

Hint: Let $S = \{u : \int_{-\infty}^\infty u(x)^2 dx = 1\}$ and let

$$F(u) = \int_{-\infty}^\infty \frac{1}{1 + x^2} u(x)^2 dx.$$

1. Show that F is continuous, for instance by proving that

$$|F(u) - F(v)| \leq 2||u - v||.$$

2. Show that $F(u) \leq 1$ for all u in S.
3. Show that there are u in S with $F(u)$ arbitrarily close to 1.
4. Show that there is no u in S with $F(u) = 1$.

Solution.